TCC Week 3. The energy space — continuation

(λ)-property: $\exists\ 0 < \lambda \in L^1_{loc}(\Omega)$, $\lambda^{-1} \in L^\infty_{loc}(\Omega)$

$$E_V(\varphi) = \int_\Omega |\nabla \varphi|^2 + V\varphi^2 \geq \int_\Omega \lambda(x)\varphi^2 \quad \forall \varphi \in C_c^\infty(\Omega)$$

Remark: $\exists\, u_* > 0$: $-\Delta u_* + V u_* \geq \underbrace{f}_{\geq 0}$ in $\Omega \Rightarrow \lambda = \frac{f}{u_*}$

Thm. If E_V satisfies (λ) then

$$\mathcal{D}^1_V(\Omega) = \text{cl}_{\|\cdot\|_V}\, C_c^\infty(\Omega), \quad \|\cdot\|_V = \sqrt{E_V(\cdot)}$$

the Hilbert space with $\langle u, v\rangle_V = \int \nabla u \nabla v + \int V uv$,

and $\mathcal{D}^1_V(\Omega) \subset L^2(\Omega, \lambda(x)dx)$.

Remark: $(\mathcal{D}^1_V(\Omega))^* \supset L^2(\Omega, x^{-1}(x)\,dx)$ Ex.

$\triangleleft$ $\ell(\varphi) := \int_\Omega f\varphi$, $f \in L^2(\Omega, x^{-1}dx)$ is a bounded functional on $\mathcal{D}^1_V(\Omega)$ $\triangleright$

Thm (Lax-Milgram thm) Assume $\mathcal{E}_V$ satisf. (x)

Then $\forall \ell \in (\mathcal{D}^1_V(\Omega))^*$ $\exists$ unique solution $u_\ell \in \mathcal{D}^1_V(\Omega)$ such that:

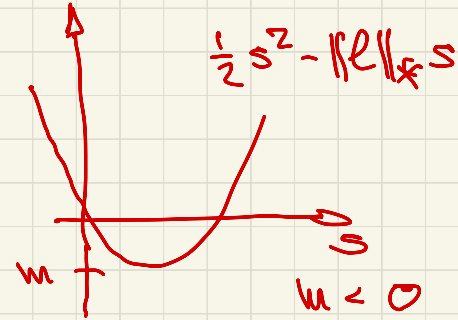
$$\langle u_\ell, \varphi \rangle_V = \ell(\varphi) \quad \forall \varphi \in \mathcal{D}^1_V(\Omega).$$

Rem. if $\ell(\varphi) = \int_\Omega f\varphi$ then $-\Delta u_\ell + V u_\ell = f$ in Ω:

$$\int_\Omega \nabla u_\ell \nabla \varphi + \int_\Omega V u_\ell \varphi = \int_\Omega f\varphi \quad \forall \varphi \in \mathcal{D}^1_V(\Omega) \supset C_0^\infty(\Omega)$$

$$\mathcal{E}(u) = \frac{1}{2}\|u\|_V^2 - \underbrace{e(u)}_{\langle e,u\rangle_V} = \frac{1}{2}\|u\|_V^2 - \langle e,u\rangle_V \geq$$

$$\geq \frac{1}{2}\|u\|_V^2 - \|e\|_* \|u\|_V \geq m$$



— $\mathcal{E}$ is Bounded Below

$$\mathcal{E}(u) \to +\infty \text{ if } \|u\|_V \to \infty$$ — coercive!

Assume $(u_n) \subset \mathcal{D}_V^1(\Omega)$ — minimising sequence

$$\mathcal{E}(u_n) \to m = \inf_{\mathcal{D}_V^1(\Omega)} \mathcal{E}$$

1) Since $\mathcal{E}$ is coercive, $\|u_n\|_V \leq M$

2) $\|u_n\|_V \leq M \implies$ converges weakly to u_0 (up to a subsequence)

3) $\mathcal{E}(u)$ is weakly lower semicontinuous,

$$u_n \rightharpoonup u_0 \implies \mathcal{E}(u_0) \leq \liminf_{n\to\infty} \mathcal{E}(u_n)$$

$\geq m = \inf \mathcal{E}$; $\liminf_{n\to\infty} \mathcal{E}(u_n) = m$

$\implies \mathcal{E}(u_0) = m$

$\implies u_0$ is a minimum of $\mathcal{E}$.

4) $\mathcal{E}(u) = \frac{1}{2}\underbrace{\|u\|_V^2}_{\text{strictly convex}} + \underbrace{\ell(v)}_{\text{linear} \Rightarrow \text{convex}}$ — strictly convex

$$\Rightarrow \mathcal{E}(\lambda u + (1-\lambda)v) < \lambda \mathcal{E}(u) + (1-\lambda)v, \quad 0 < \lambda < 1.$$

Ex. $\|u\|^2$ is strictly convex.

$\Rightarrow$ u_0 is unique minimiser

(Otherwise $\mathcal{E}(\lambda u_0 + (1-\lambda)v_0) < \lambda m + (1-\lambda)m = m$)

5) Since u_0 is the minimiser,

$$\underbrace{\langle u_0, \varphi \rangle = \ell(\varphi)}_{\text{Euler-Lagrange eq.}} \quad \forall \varphi \in \mathcal{D}_V^1(\Omega)$$

1) $\mathcal{D}^1_0(\mathbb{R}^N)$, $N \geq 3$ – is the energy space

for $\int_{\mathbb{R}^N} |\nabla u|^2$ $\left(\lambda(x) = (1+|x|^2)^{-\frac{N-2}{2}} \right)$

$\Rightarrow$ $-\Delta u = f$ in $\mathbb{R}^N$ has unique solution

$\forall f \in L^2\left(\mathbb{R}^N, (1+|x|^2)^{\frac{N-2}{2}}\right) \not\subset L^2(\mathbb{R}^N)$

2) $H^1(\mathbb{R}^N)$, $\int |\nabla u|^2 + \int u^2 \geq \int |\nabla u|^2 - \lambda$!

$H^1(\mathbb{R}^N) \subset \mathcal{D}^1_0(\mathbb{R}^N)$

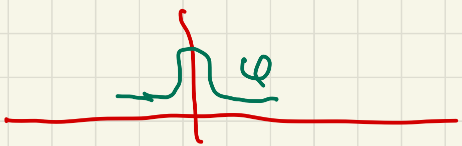
$H^1(\mathbb{R}^N) = \mathcal{D}^1_0(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$

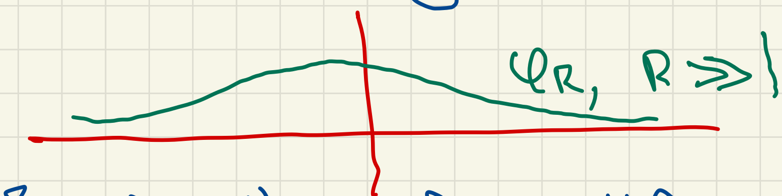
$\Rightarrow (H^1(\mathbb{R}^N))^* \supset L^2(\mathbb{R}^N)$

3) $\int_{\mathbb{R}^N} |\nabla u|^2 - \int_{\mathbb{R}^N} u^2$ — $\geq$? — no!

Scaling arguement. Take $\varphi \in C_0^\infty(\mathbb{R}^N)$

$$\varphi_R(x) = \varphi\left(\frac{x}{R}\right)$$ – rescaling of φ





$$\int_{\mathbb{R}^N} \varphi_R^2(x)\,dx = \int_{\mathbb{R}^N} \varphi^2\left(\frac{x}{R}\right) R^N d\left(\frac{x}{R}\right) = R^N \int_{\mathbb{R}^N} \varphi^2$$

$$\int_{\mathbb{R}^N} |\nabla \varphi_R(x)|^2 = \int \left|\nabla \varphi\left(\frac{x}{R}\right)\right|^2 = \int \left|\frac{1}{R}\nabla\varphi\right|^2 R^N \, d\left(\frac{x}{R}\right)$$

$$= R^{N-2} \int |\nabla \varphi|^2$$

$$\int |\nabla \varphi_R|^2 - \int |\varphi_R|^2 = R^{N-2} \underbrace{\int |\nabla\varphi|^2}_{A} - R^N \underbrace{\int \varphi^2}_{B} =$$

$$= A R^{N-2} - B R^N \to -\infty \quad \text{as } R \to \infty$$

$$\Rightarrow \boxed{\inf \sigma(-\Delta) = 0}$$

(not needed for us)

Ex. Ω — Bounded domain.

Then $\int |\nabla \varphi|^2 \geq \lambda_1 \int \varphi^2 \quad \forall \varphi \in C_0^\infty(\Omega)$

$\lambda_1 = \lambda_1(\Omega) > 0$

$\Rightarrow \int |\nabla u|^2 - \lambda \int u^2 \geq (\lambda_1 - \lambda) \int u^2$ — $\textcircled{A}$ $\forall \lambda < \lambda_1$

$(\lambda_1 - \lambda) > 0 \quad \forall \lambda < \lambda_1$

Remark: If $\lambda = \lambda_1 \Rightarrow \int |\nabla \varphi_1|^2 - \lambda_1 |\varphi_1|^2 = 0$,

$\varphi_1 > 0$ — principal eigenvalue!

According to Agmon and Pinchover

1) $-\Delta + V$ is subcritical if it satisfies (a)
(think of $-\Delta - \lambda$ on Ω with $\lambda < \lambda_1$)

2) $-\Delta + V$ is critical if (a) fails but
$\int |\nabla \varphi|^2 \geq \int V \varphi \quad \forall \varphi \in C_0^\infty(\Omega)$
(think of $-\Delta - \lambda_1$ on Ω or $-\Delta - \frac{c_H}{|x|^2}$ on $\mathbb{R}^n$)

3) $-\Delta + V$ is supercritical if $\exists \varphi \in C_c^\infty(\Omega)$
$\int |\nabla \varphi|^2 + \int V \varphi^2 < 0$.

Example: $N \geq 3$

$$\int_{\mathbb{R}^N} |\nabla \varphi|^2 - \int_{\mathbb{R}^N} \frac{C_H}{|x|^2} \varphi^2 \geq \frac{1}{4} \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2 \log^2 |x|}$$

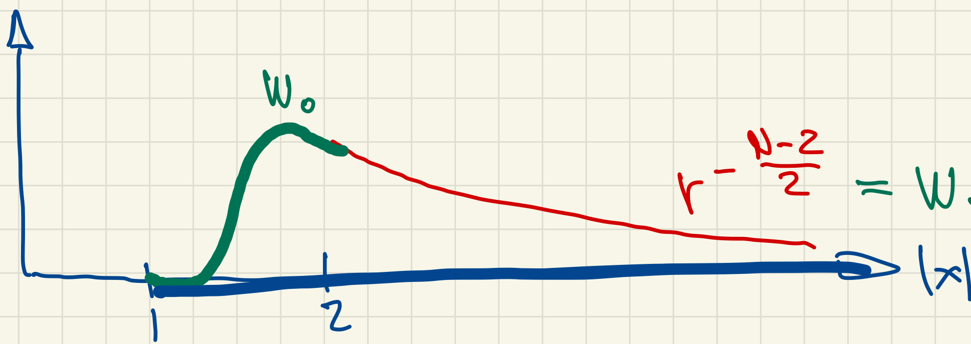
(λ) $\forall \varphi \in C_0^\infty(\mathbb{R}^N \setminus \overline{B_1})$

Consider $\mathcal{D}^1_{\frac{C_H}{|x|^2}}(\mathbb{R}^N \setminus \overline{B_1})$

Take

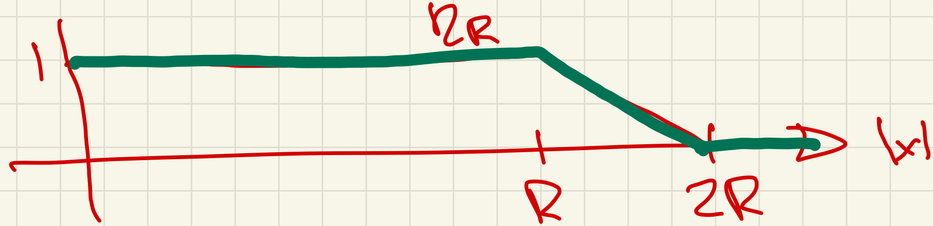
$$w(x) = \begin{cases} |x|^{-\frac{N-2}{2}}, & |x| > 2 \\ w_0(x), & 2 < |x| < 1 \end{cases}$$

w_0 is such that w is C^2, $w \geq 0$ and $w(x) = 0$, $|x| < 1\frac{1}{2}$



$r^{-\frac{N-2}{2}} = w_*$ if $r > 2$

$$w_R = \eta_R\, w \;, \qquad \text{(comp. supp.)}$$



$$\left(-\Delta - \frac{c_H}{|x|^2}\right) \underbrace{r^{-\frac{N-2}{2}}}_{w_*} = 0 \qquad \text{AAP!}$$

$$\int |\nabla w_R|^2 - \int \frac{c_H}{|x|^2} w_R^2 \underset{\text{AAP}}{=} \int \left|\nabla \frac{w_R}{w_*}\right|^2 w_*^2 =$$

$$= \underbrace{\int_1^2 \left|\nabla \frac{w_R}{w_*}\right|^2 w_*^2}_{c_1} + \int_2^{2R} \underbrace{\left|\nabla \frac{\eta_R w_*}{w_*}\right|^2}_{\frac{1}{R}} w_*^2 =$$

$$\approx c_1 + \frac{c_2}{R^2} \int_R^{2R} \left(r^{-(N-2)}\right) r^{N-1}\, dr =$$

$$= c_1 + c_2 \frac{1}{R^2} \left(\log 2R - \log R \right) =$$

$$= c_1 + c_2 \frac{\log 2}{R^2} \xrightarrow[R\to\infty]{} c_1$$

$$\Rightarrow \boxed{w \in \mathcal{D}'_{\frac{c_H}{|x|^2}}\left(\mathbb{R}^N \setminus \overline{B}_1\right)}$$

$$E_V(w) = \int_{\mathbb{R}^N} |\nabla w|^2 - \int_{\mathbb{R}^N} \frac{c_H}{|x|^2} w^2 = c_1$$

But $\underbrace{\int |\nabla w_R|^2}_{\to +\infty} - \underbrace{\int \frac{c_H}{|x|^2} w_R^2}_{\to -\infty}$,

so $\int |\nabla w|^2 - \int \frac{c_H}{|x|^2} w^2$ is not defined!

Yet $E_V(w) = c_1 < \infty$, well defined

$w \in \mathcal{D}^1_{\frac{c_H}{|x|^2}}(\mathbb{R}^N \setminus \overline{B_1})$ but $w \notin \mathcal{D}^1_0(\mathbb{R}^N \setminus \overline{B_1})$